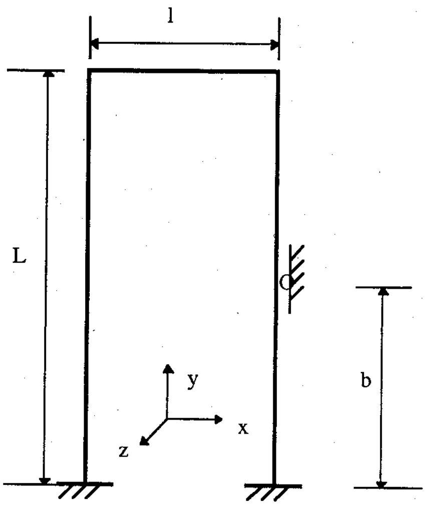

Fig. 2 Plane frame with an in-span support.

angular plane frame with an in-span support. Now let us assume that the plane frame has the physical properties: Young's modulus $= 2.1 \times 10^{11}$ N/m^2, mass density $= 7.8 \times 10^3$ kg/m^3, cross-sectional area $= 0.10 \times 0.10 = 0.01$ m^2, Poisson's rate $= 0.3$, $L = 12$ m, $l = 4$ m, $b = 6.00$ m.

The first four natural frequencies and the corresponding mode shapes are obtained using finite element method (FEM) by dividing the plane frame uniformly into 56 beam elements. Note that the mode shapes are assumed to be normalized to unit modal mass. Since for a beam element via FEM the slope of mode shapes $\hat{y}'(b)$ at a grid point is just the available rotation in Z sense of that grid point, the rate of eigenvalues with respect to b can be evaluated directly using Eqs. (13) and (14). The results are summarized in Table 1.

For the purpose of comparison, the rate of eigenvalues are also computed using forward finite difference (FFD) method with $\Delta b = 0.01$ and the results are included in the fourth column of Table 1. The FFD method used here can be stated as

$$\frac{d\omega_i^2}{db} = \frac{\omega_i^2(b + \delta b) - \omega_i^2(b)}{\delta b} \tag{15}$$

As can be seen in the last column of Table 1, the results obtained using Eqs. (13) and (14) agree well with the results obtained using the FFD method.

V. Concluding Remarks

A new method for deriving the formulas of eigenvalue rate with respect to in-span support location is presented in this Note. Based on the generalized variational principle, the approach presented can be easily extended to include the continuum and other structural members. The eigenvalue rate with respect to in-span support location can be used to find desirable locations of supports to maximize the fundamental natural frequencies of a structure member or can be used for the purpose of reanalysis of the structure with modified support locations.

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New Method for the Improvement of Measured Modes Through Orthogonalization

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Introduction

FOR large space structures, the mode shapes obtained from tests seldom have acceptable compatibilities with the analytically derived model. Generally, the orthogonality check of the measured modes by means of the analytical mass matrix results in some off-diagonal elements that cannot be ignored. We present a new method to perform the orthogonalization of the measured modes, assuming that the analytical mass matrix is exactly known. It is noted that the selection of the mass matrix as a reference base in the orthogonalization process is not unique; the problem of satisfying orthogonality has been attacked from different directions: Some methods[1–5] consider that the measured modes are more reliable and use them to adjust the analytical mass matrix to achieve orthogonality. Others[6–8] consider that the analytically derived mass matrix is correct and adjust the measured modes so that the orthogonalization condition is satisfied. After the orthogonalization is achieved, the analytical stiffness matrix can also be improved. Discussions on the degree of applicability of the two distinct approaches in the orthogonalization process can be found in Refs. 9 and 10.

For the orthogonalization of the measured modes in this Note, we seek a solution in an optimal way following the methodology of Baruch and Bar Itzhack.[8] However, we do not impose any restrictions on the type of the transformation matrix, triangular or symmetric, and thus the resulting adjusted modes are closest to the measured ones. Higher confidences to the lower modes can be considered by a diagonal weighting matrix. The adjusted modes are orthogonal to the rigid-body ones, if any. A simple closed-form solution, as well as an iterative procedure, is derived and demonstrated with an example; a comparison with the Targoff/Baruch method[7,8] is also presented.

Method Description

The problem can be mathematically expressed as follows: Given a mass matrix of a structure M $(n \times n)$, a rigid-body modal matrix ϕ_r $(n \times r)$ that is orthogonal with respect to the mass matrix, and a measured modal matrix ϕ_m $(n \times m)$, $m < n$, determine an adjusted flexible modal matrix ϕ $(n \times m)$ that satisfies the orthogonality condition:

$$\phi_r^t M \phi = 0 \tag{1}$$

$$\phi^t M \phi = I \tag{2}$$

where the superscript t denotes matrix transpose, and I represents a unit matrix. Assuming

$$\phi = [\phi_r \phi_m] \begin{bmatrix} R \\ C \end{bmatrix} \tag{3}$$

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where $[^R_C]$ is the transformation matrix, and substituting it into Eq. (1), we obtain

$$R = -(\phi_r^t M \phi_m) C \tag{4}$$

Substitution of Eqs. (3) and (4) into Eq. (2) leads to

$$C^t A C = I \tag{5}$$

where

$$A = \begin{bmatrix} -\phi_r^t M \phi_m \\ I \end{bmatrix}^t [\phi_r \phi_m]^t M [\phi_r \phi_m] \begin{bmatrix} -\phi_r^t M \phi_m \\ I \end{bmatrix} \tag{6}$$

The error function to be minimized is defined as

$$\varepsilon = ||(\phi - \phi_m)w|| = ||(\psi C - \phi_m)w|| = \sum_i^n \sum_j^m [(\psi C - \phi_m)w]_{ij}^2 \tag{7}$$

where

$$\psi = [\phi_r \phi_m] \begin{bmatrix} -\phi_r^t M \phi_m \\ I \end{bmatrix} \tag{8}$$

and w is a diagonal weighting matrix, which, if required, assigns higher credibility to the lower measured modes. Now we define the Lagrange function as

$$L = ||(\psi C - \phi_m)w|| + \sum_i^m \sum_j^m \lambda_{ij}[C^t A C - I]_{ij} \tag{9}$$

where λ_{ij} are the Lagrange multipliers, and due to the symmetry of Eq. (5), $\lambda_{ji} = \lambda_{ij}$. By setting the partial derivatives of L with respect to matrix C to zero, we obtain

$$2\psi^t \psi C w^2 - 2\psi^t \phi_m w^2 + 2AC\Lambda = 0 \tag{10}$$

Closed Form Solution

If $w = I$, the problem can be solved in closed form. Substituting Eq. (5) into Eq. (10), and noting that $\Lambda^t = \Lambda$, we have

$$C^t \psi^t \phi_m = \phi_m^t \psi C \tag{11}$$

It is seen from Eq. (11) that matrix C can be expressed by

$$C = (\phi_m^t \psi)^{-1} D \tag{12}$$

where D is a symmetric matrix to be determined. Substitution of Eq. (12) into Eq. (5) yields

$$D^t (\psi^t \phi_m)^{-1} A (\phi_m^t \psi)^{-1} D = I \tag{13}$$

Equation (13), from theory of linear algebra, implies that $(\psi^t \phi_m)^{-1} A (\phi_m^t \psi)^{-1}$ is a positive definite matrix, and thus we have

$$D = \left[(\psi^t \phi_m)^{-1} A (\phi_m^t \psi)^{-1}\right]^{-\frac{1}{2}} = \sum_i^m \mu_i^{-\frac{1}{2}} P_i P_i^t \tag{14}$$

with μ_i and P_i indicating the ith eigenvalue and eigenvector of $(\psi^t \phi_m)^{-1} A (\phi_m^t \psi)^{-1}$, respectively. Finally, the adjusted modal matrix ϕ is obtained from Eqs. (3), (4), (12), and (14) through simple substitution.

Table 1 Transformation matrix (present method)

1.0597	−0.0143	0.0209	−0.1582	0.1542
0.0903	1.0298	0.0492	−0.0288	−0.1785
0.0157	0.0442	1.0040	0.0563	0.0139
−0.1560	−0.0487	−0.0094	1.0359	0.0188
0.1084	−0.1097	0.0165	−0.0583	1.0633

Table 2 Transformation matrix (Targoff/Baruch)

1.0623	0.0337	0.0147	−0.1535	0.1376
0.0337	1.0384	0.0464	−0.0465	−0.1415
0.0147	0.0464	1.0052	0.0241	0.0145
−0.1535	−0.0465	0.0241	1.0361	−0.0181
0.1376	−0.1415	0.0145	−0.0181	1.0577

Iterative Algorithm of Solution

If $w \neq I$, the solution can be achieved by an iterate procedure. In this case, Eq. (11) becomes

$$C^t (\psi^t \psi C - \psi^t \phi_m) w^2 = w^2 (C^t \psi^t \psi - \phi_m^t \psi) C \tag{15}$$

The following iterative equation is constructed to incorporate Eqs. (5) and (15):

$$C_{n+1} = C_n X_n \tag{16}$$

where

$$X_n = C_n^t A C_n + \Delta_1 \left[C_n^t (\psi^t \psi C_n - \psi^t \phi_m) w^2 - w^2 (C_n^t \psi^t \psi - \phi_m^t \psi) C_n\right] \tag{17}$$

where Δ_1 is a scalar adjust factor. It is recommended that $\Delta_1 = 0.2/\max(w_{ii}^2)$ so that the magnitudes of the two terms in X_n are comparable.

If Eq. (16) converges and $|C_\infty| \neq 0$, then $X_\infty = I$. Since the first term of X_n, $C_n^t A C_n$, is symmetric, the second term will also be symmetric in the limit; this second term then vanishes and enforces the equality of Eq. (15). Therefore, $X_\infty = C^t A C = I$.

Example

The iterative procedure has been successfully applied to several numerical examples. The example analyzed herein was initially presented by Baruch and Bar Itzhack.[8] The structure is a lifting surface of a flight vehicle modeled by a 37-degree-of-freedom lumped model. Only five modes were measured and used in the orthogonalization process. The maximum value of the orthogonality check of the measured modes is more than 28%.[8] The mass matrix and measured modes are given in Ref. 8.

The transformation matrices for the present method and the Targoff/Baruch method are presented in Tables 1 and 2, respectively. The transformation matrices determined by the two approaches are comparable; the unnecessary constraint of symmetry in the transformation matrix, however, is waived in the present approach. The change in the adjusted modes, defined by $||\phi - \phi_m||$, for the present method is smaller than that of Targoff/Baruch (0.6488 vs 0.7411). The iterative procedure converges very fast; after only four iterations, the maximum absolute value of off-diagonal elements in the orthogonality check is less than 0.5%.

Conclusions

A new procedure for the optimal orthogonalization of incomplete measured modes has been described. The approach achieves the smallest changes in the measured modes compared with other methods in the literature. A closed-form solution as well as an iterative algorithm are derived. The example presented also illustrates the fast convergence of the iterative algorithm.

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